§ 15.4/15.5 Triple Integrals

- The triple integral:

$$\iiint_D f(x,y,z)\,dV$$

We'd like to call this "the 4-D volume under the graph of f 'above' the 3-D volume D"

But we can't visualize 4-D volumes —

So we view it as

"A sum of weighted 3-volumes"

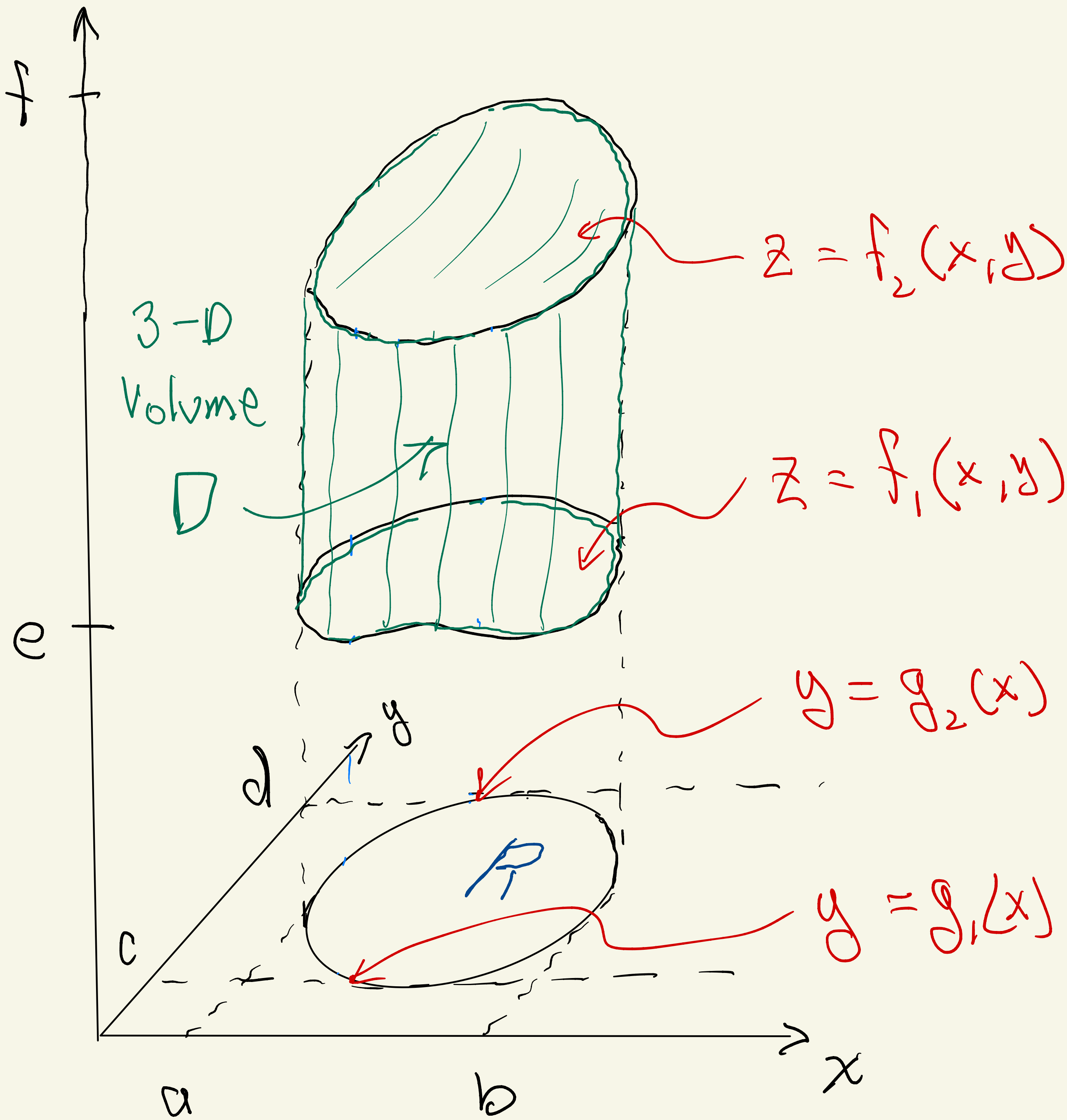
- For this - define the integral as the limit of a 3-D Riemann Sum: $\iiint_D f(x,y,z)\,dV$

$$= \lim_{N\to\infty} \sum_{(x_i, y_j, z_k)\in D} \overbrace{f(x_i, y_j, z_k)\underbrace{\Delta x \Delta y \Delta z}_{\Delta V_{ijk}}}^{\text{weighted 3-volume}}$$

$\underbrace{\qquad\qquad}_{\text{3-D Riemann Sum}}$

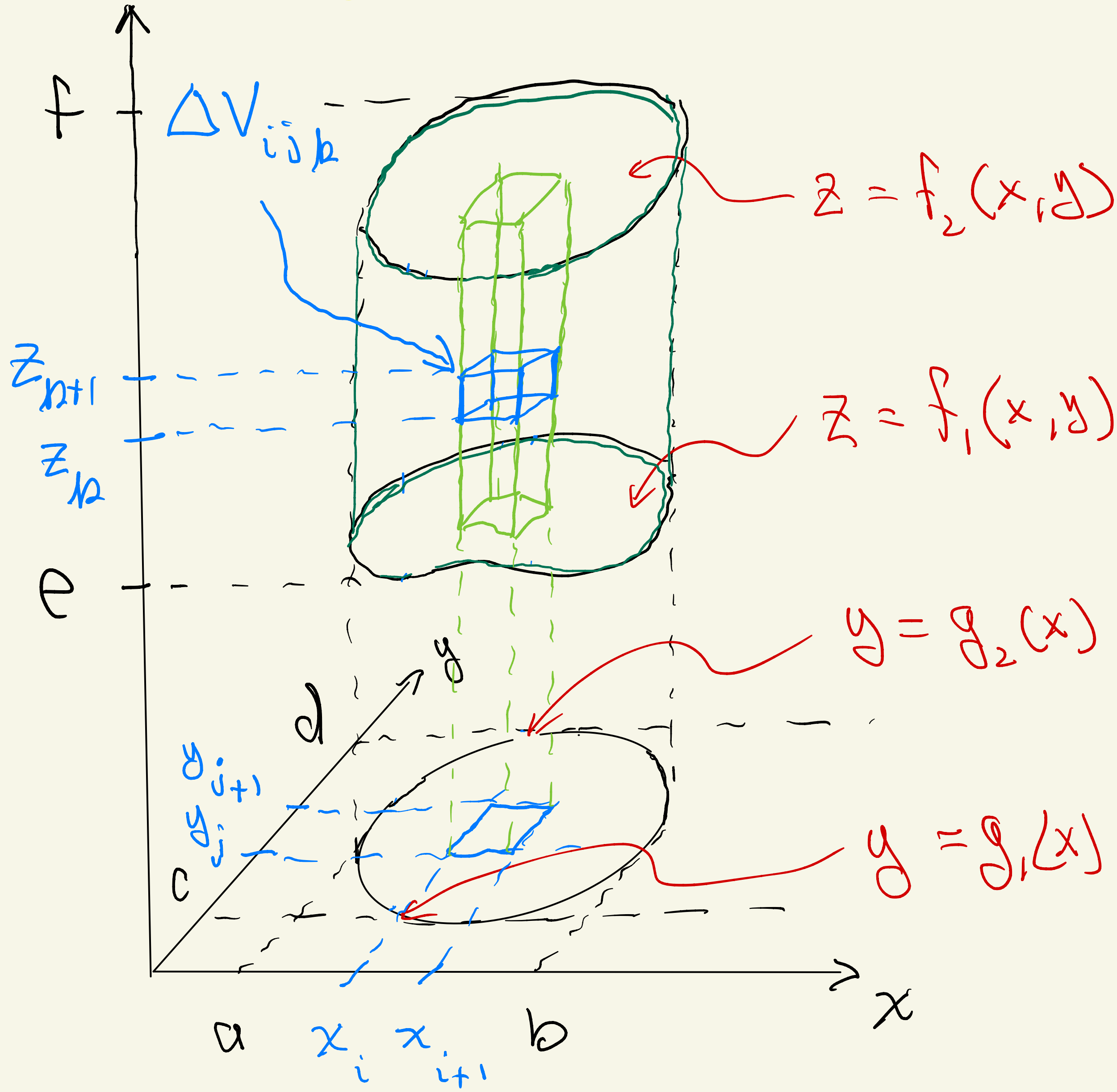
- Approx by 4-D rectangles and view the integral as "a sum of weighted 3-volumes"

- The basic Picture:



$$D \subseteq [a,b] \times [c,d] \times [e,f]$$

3-D Riemann Sum

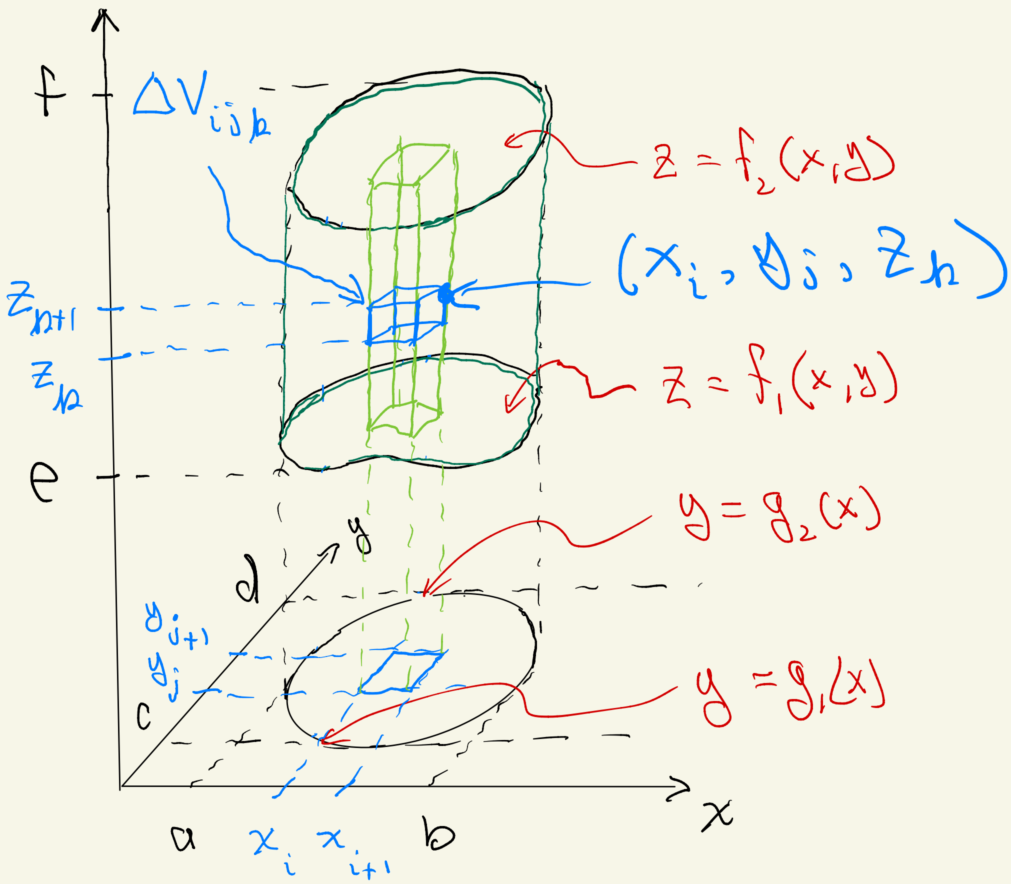


$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_N = b$$

$$c = y_0 < y_1 < y_2 < \cdots < y_j < \cdots < y_N = d$$

$$e = z_0 < z_1 < \cdots < z_N = f$$

$$\Delta x = \frac{b-a}{N}, \quad \Delta y = \frac{d-c}{N}, \quad \Delta z = \frac{f-e}{N}$$



$$\Delta x = \frac{b-a}{N}, \quad \Delta y = \frac{d-c}{N}, \quad \Delta z = \frac{f-e}{N}$$

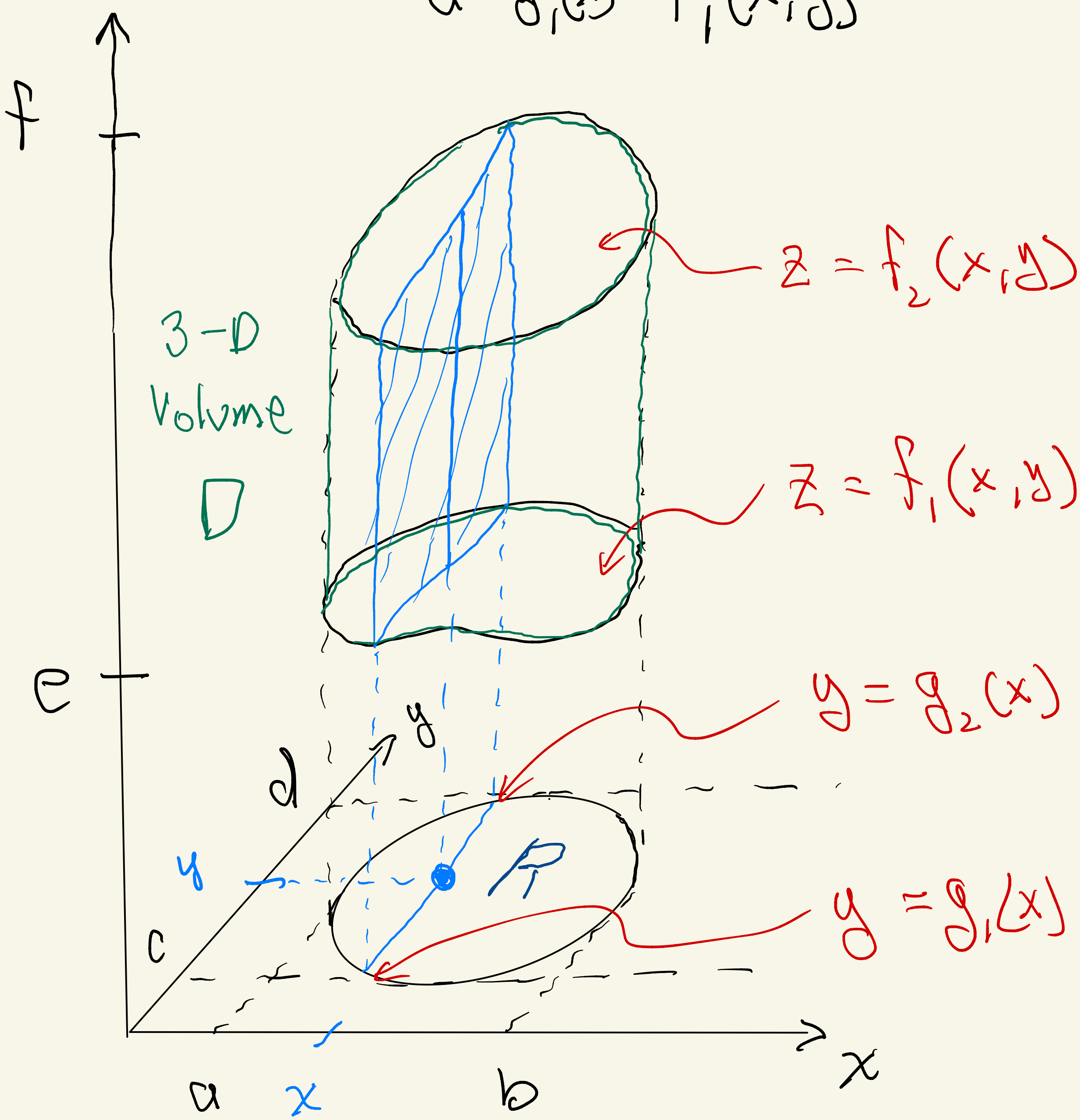
- Definition of 3-D integral as a Riemann Sum:

$$\iiint_D f(x,y,z)\,dV = \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j,z_k)\in D} f(x_i,y_j,z_k)\,\Delta x\,\Delta y\,\Delta z}_{\text{Approximation by Riemann Sum}}$$

Approximation by Riemann Sum

• Iterate the Integral to get exact value

$$\iiint\limits_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



§15.5 Same idea to get Area / Mass / Center of Mass / Moments of Inertia formulas from Riemann Sum?

Assume D is a 3-D object with density $\delta(x,y,z) = \dfrac{\text{mass}}{\text{volume}}$

- **Volume:** Take $\delta = 1$

$$\text{Vol}(D) = \lim_{N \to \infty} \sum_{(x_i, y_j, z_k) \in D} 1 \cdot \Delta x \Delta y \Delta z$$

$$= \lim_{N \to \infty} \sum_{D} \Delta V_{ijk}$$

$$V = \iiint_D dV$$

8

- **Mass:**

$$\text{Mass}(D) = \lim_{N \to \infty} \sum_{(x_i, y_j, z_k) \in D} \delta(x_i, y_j, z_k) \Delta x \Delta y \Delta z$$

$$\frac{\Delta \text{Mass}}{\Delta \text{Vol}} \cdot \text{Vol}$$

$$= \lim_{N \to \infty} \sum_{D} \Delta M_{ijk}$$

$$M = \iiint_D \delta(x, y, z)\, dV$$

- Center of Mass (Similarly)

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Eg $$M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$$

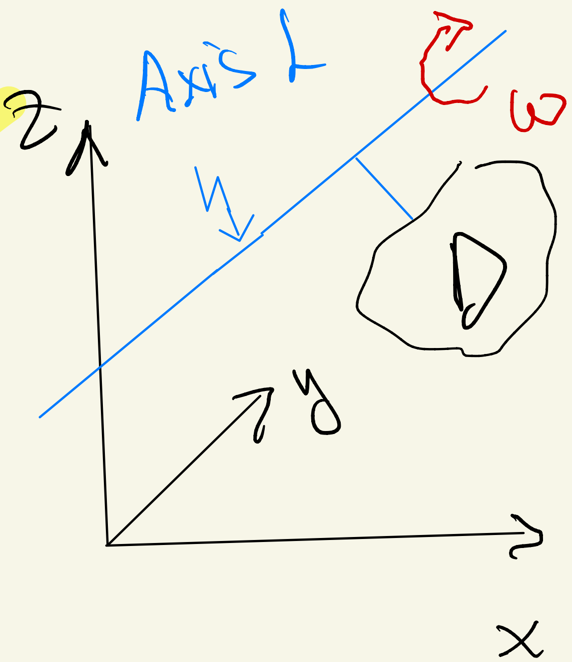
Distance to yz-plane = x

$$M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$$

$$M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$$

• Kinetic Energy of Rotation

Q: What is the KE of rotation about axis L at angular velocity $\omega = \frac{d\theta}{dt}$?



$$KE = \lim_{N\to\infty} \sum_{D} \frac{1}{2} \Delta M_{ijk} \left(\omega r_{ijk}\right)^2$$

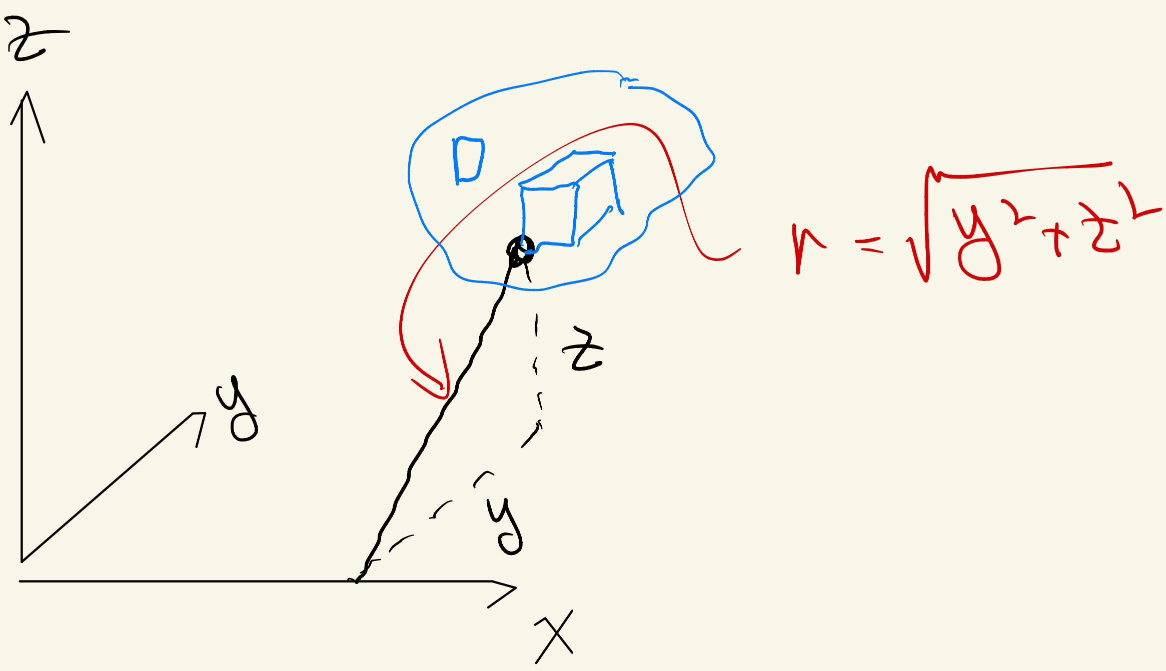
$$= \frac{1}{2}\omega^2 \lim_{N\to\infty} \sum_{D} r_{ijk}^2 \, \delta(x_i, y_j, z_k) \, \Delta x \Delta y \Delta z$$

$$= \frac{1}{2}\omega^2 \underbrace{\iiint_D r(x,y,z)^2 \, \delta(x,y,z) \, dV}_{I_L} = \frac{1}{2} I_L \omega^2$$

Conclude:

$$KE = \frac{1}{2} I_L \omega^2$$

Eg $$I_x = \iiint_D \underbrace{(y^2+z^2)}_{\text{distance to } x\text{-axis}} \delta(x,y,z)\, dV$$



Radius of Gyration R

is defined by the equation

$$\frac{1}{2} M \omega^2 R^2 = \frac{1}{2} I_L \omega^2$$

Solve for R:

$$\frac{1}{\cancel{2}} M \cancel{\omega^2} R^2 = \frac{1}{\cancel{2}} I_L \cancel{\omega^2}$$

$$R^2 = I_L / M$$

$$R = \sqrt{\frac{I_L}{M}}$$

Examples

1. Find the volume of the tetrahedron D bounded from above by the plane thru $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, and on the sides by the xy-plane, yz-plane, xz-plane

Assume: $a, b, c > 0$ positive nv

- Draw the region

Eq for plane: $z = Ax + By + C$

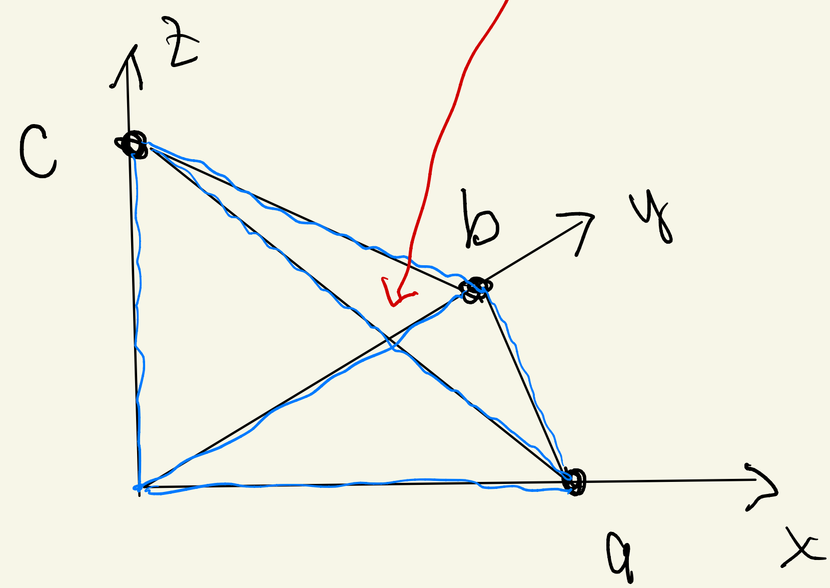
$(0,0,c) \Rightarrow c = 0 + 0 + C$ $\boxed{C = c}$

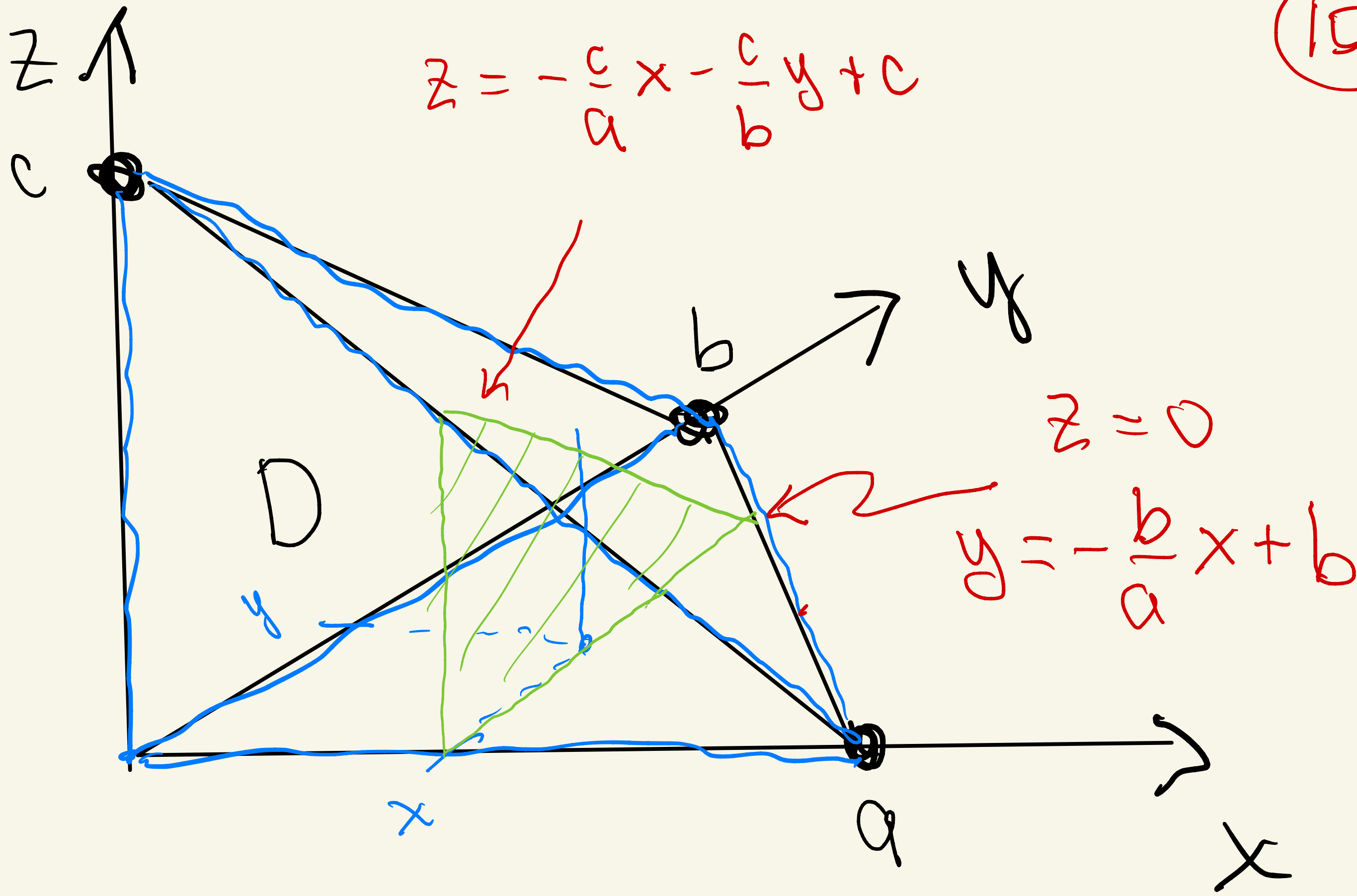
$(0,b,0) \Rightarrow 0 = 0 + Bb + c$ $\boxed{B = -\frac{c}{b}}$

$(a,0,0) \Rightarrow 0 = Aa + c$ $\boxed{A = -\frac{c}{a}}$

$$\boxed{z = -\frac{c}{a}x - \frac{c}{b}y + c}$$

- From the picture we get limits of integration





$$\text{Vol} = \iiint_D 1 \cdot dV$$

$$= \int_0^a \underbrace{\int_0^{-\frac{b}{a}x+b} \underbrace{\int_0^{-\frac{c}{a}x-\frac{c}{b}y+c} dz}_{\text{fix } (x,y)} \, dy}_{\text{fix } x} \, dx$$

Evaluate by 3 Math 21B integrals

$$\mathrm{Vol}(D) = \int_0^a \int_0^{-\frac{b}{a}x+b} \underbrace{\int_0^{-\frac{c}{a}x-\frac{c}{b}y+c} dz}\, dy\, dx$$

$$z\Big]_0^{*} = -\frac{c}{a}x - \frac{c}{b} + c$$

$$= \int_0^a \underbrace{\int_0^{-\frac{b}{a}x+b} -\frac{c}{a}x - \frac{c}{b}y + c \, dy}\, dx$$

$$-\frac{c}{a}xy - \frac{c}{b}\frac{y^2}{2} + cy\Big]_0^{-\frac{b}{a}x+b}$$

$$-\frac{c}{a}x\left(-\frac{b}{a}x+b\right) - \frac{c}{b}\frac{1}{2}\left(-\frac{b}{a}x+b\right)^2$$

$$+c\left(-\frac{b}{a}x+b\right)$$

Have:

$$= \int_0^a \underbrace{\int_0^{-\frac{b}{a}x+b} -\frac{c}{a}x - \frac{c}{b}y + c \, dy}_{-\frac{c}{a}x\left(-\frac{b}{a}x+b\right) - \frac{c}{b}\frac{1}{2}\left(-\frac{b}{a}x+b\right)^2 + c\left(-\frac{b}{a}x+b\right)} \, dx$$

$$= \int_0^a \frac{cb}{a^2}x^2 - \frac{cb}{a}x - \frac{cb}{a}x + bc - \frac{1}{2}\frac{c}{b}\left(\frac{b^2}{a^2}x^2 - 2\frac{b^2}{a}x + b^2\right) dx$$

$$= \int_0^a \frac{cb}{a^2}x^2 - \cancel{2}\frac{cb}{a}x + bc - \frac{1}{2}\frac{cb}{a^2}x^2 + \cancel{\frac{cb}{a}}x - \frac{1}{2}cb$$

$$= \int_0^a \frac{1}{2}\frac{bc}{a^2}x^2 - \frac{bc}{a}x + \frac{1}{2}bc \quad dx$$

$$= \frac{bc}{a}\int_a^b \frac{1}{2}\frac{x^2}{a} - x + \frac{1}{2}a \, dx$$

$$= \int_0^a \left[\frac{1}{2}\frac{bc}{a^2}x^2 - \frac{bc}{a}x + \frac{1}{2}bc\right]dx$$

$$= \frac{bc}{a}\int_a^b \left[\frac{1}{2}\frac{x^2}{a} - x + \frac{1}{2}a\right]dx$$

$$= \frac{bc}{a}\left[\frac{1}{6}\frac{x^3}{a} - \frac{x^2}{2} + \frac{1}{2}ax\right]_0^a$$

$$= \frac{bc}{\cancel{a}}\left[\frac{1}{6}\frac{a^{\overset{1\,\cancel{2}}{\cancel{3}}}}{\cancel{a}} - \frac{a^{\overset{1}{\cancel{2}}}}{2} + \frac{1}{2}a^{\overset{1}{\cancel{2}}}\right]$$

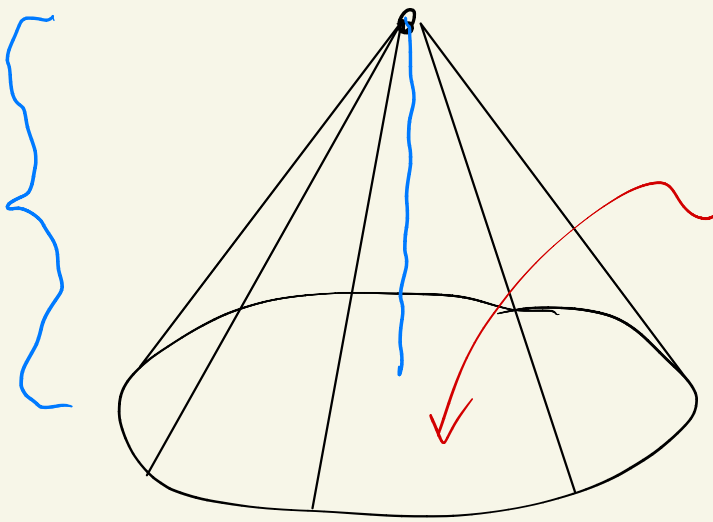
$$= bc\left[\frac{1}{6}a - \cancel{\frac{1}{2}a} + \cancel{\frac{1}{2}a}\right]$$

$$= \frac{1}{6}abc$$

Volume of Tetrahedron $V = \frac{abc}{6}$

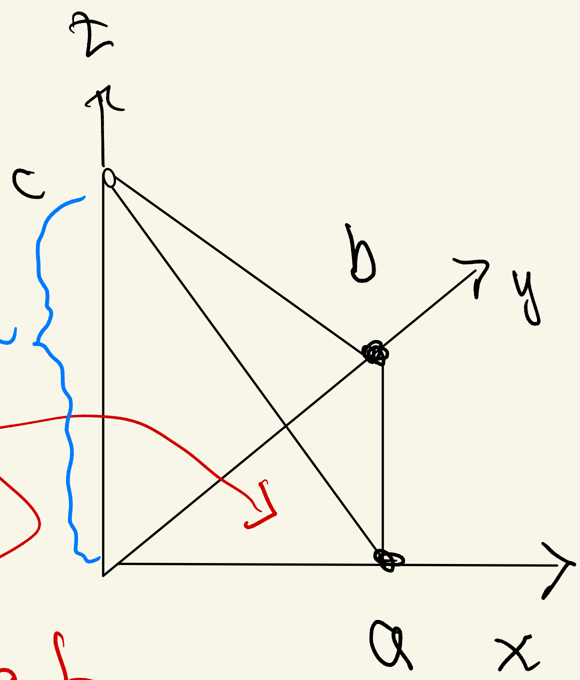
Check:

The volume of a cone:



$$\text{Volume} = \frac{1}{3} A h$$

For Tetrahedron



$$\text{Vol} = \frac{1}{3} A h$$

$$= \frac{1}{3}\left(\frac{1}{2} ab\right) c = \frac{abc}{3} \checkmark$$

Area base $= \frac{1}{2} ab$

Main Take Away: The essence (20) of integration in 3-dimensions is captured in the following picture